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Uniform estimates for polynomial approximation in domains with corners

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Abstract

Let $G \subset \mathbb{C}$ be a domain with a Jordan boundary ∂G, consisting of l smooth curves Γ_j, such that $\{z_j\} := \Gamma_{j-1} \cap \Gamma_j \neq \emptyset$, $j = 1, \ldots, l$, where $\Gamma_0 := \Gamma_l$. Denote by $\alpha_j \pi$, $0 < \alpha_j \leqslant 2$, the angles at z_j's between the curves Γ_{j-1} and Γ_j, exterior with respect to G. Let Φ be a conformal mapping of the exterior $\mathbb{C} \setminus \overline{G}$ of $\overline{G} = G \cup \partial G$ onto the exterior of the unit disk, normed by $\Phi'(\infty) > 0$. We assume that there is a neighborhood U of $\overline{G}$, such that $0 < c(G) \leqslant \varphi(z) \left|\Phi'(z)\right| \leqslant C(G), \quad z \in U \setminus \overline{G}$, where

$$\varphi(z) := \prod_{j=1}^{l} \left|z - z_j\right|^{1 - \frac{1}{\alpha_j}}, \quad z \in \mathbb{C},$$

$z \neq z_j$ if $\alpha_j \leqslant 1$. Set $\|g\|_G := \sup\{|g(z)| : z \in G\}$. Then we prove Theorem. Let $r \in \mathbb{N}$ and $0 \leqslant \beta \leqslant r$. If a function f is analytic in G and $\left\| f^{(r)} \varphi^{\beta} \right\|_G < +\infty$, then for each $n \geqslant lr$ there is an algebraic polynomial P_n of degree $< n$, such that

$$\|(f - P_n)\varphi^{\beta - r}\|_G \leqslant \frac{c(r, G)}{n^r} \|f^{(r)} \varphi^{\beta}\|_G.$$

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1. Definitions and main results

Let $G \subset \mathbb{C}$ be a domain with a Jordan boundary ∂G, consisting of l smooth curves Γ_j, such that $\{z_j\} := \Gamma_{j-1} \cap \Gamma_j \neq \emptyset$, $j = 1, \ldots, l$, where $\Gamma_0 := \Gamma_l$. Denote by $\alpha_j \pi$, $0 < \alpha_j \leqslant 2$, the angles at z_j's between the curves Γ_{j-1} and Γ_j, exterior with respect to the domain G. Set $\overline{G} := G \cup \partial G$, the closure of G.

For a function $g : G \to \mathbb{C}$ let

$$\|g\|_G := \sup_{z \in G} |g(z)|$$

be its sup norm, which may be finite and infinite. To unify formulations we will assume that $+\infty \leqslant +\infty$. Let $\mathbb{P}_n$ be the space of algebraic polynomials of degree $\leqslant n-1$. We denote by

$$E_n(g, G) := \inf_{P_n \in \mathbb{P}_n} \|g - P_n\|_G ,$$

the error of the best polynomial approximation of g. Let Φ be a conformal mapping of the exterior $\mathbb{C} \setminus \overline{G}$ of $\overline{G}$ onto the exterior of the unit disk, normed by $\Phi'(\infty) > 0$. We apply Dzjadyk's classical methods of the constructive theory of the approximation of functions on the sets of the complex plane. To this end we assume that there is a neighborhood U of $\overline{G}$, such that

$$c \leqslant \varphi(z) \left|\Phi'(z)\right| \leqslant C, \quad z \in U \setminus \overline{G}, \tag{1.1}$$

where $c = c(G)$ and $C = C(G)$ are positive constants, that depend only on G, and

$$\varphi(z) := \prod_{j=1}^{l} \left|z - z_j\right|^{1-\frac{1}{\alpha_j}}, \quad z \in \mathbb{C},$$

$z \neq z_j$ if $\alpha_j \leqslant 1$. To satisfy (1.1) one should require that all l smooth curves Γ_j, constituting the boundary ∂G, must be "a little more than smooth". Say, they may be Ljapunov curves, or even less smooth than Ljapunov curves, so-called Dini-type curves; see [2,14].

Everywhere below we denote by c different positive constants that may depend only on G and $r \in \mathbb{N}$.

The first result we present is as follows:

Theorem 1. *Let $r \in \mathbb{N}$. If a function f is analytic in G, then*

$$E_n(f, G) \leqslant \frac{c}{n^r} \left\| f^{(r)} \varphi^r \right\|_G , \quad n \geqslant r.$$

Theorem 1 is the analog of the well-known estimate of approximation on the interval $[-1, 1]$; see Babenko [4], Ditzian and Totik [8].

We also have

Theorem 2. *Let $r \in \mathbb{N}$. If a function f is analytic in G, then for each $n \geqslant lr$ there exists a polynomial $P_n \in \mathbb{P}_n$, such that*

$$\left\| \frac{f - P_n}{\varphi^r} \right\|_G \leqslant \frac{c}{n^r} \left\| f^{(r)} \right\|_G .$$

Theorem 2 is close to Dzjadyk's classical direct theorem and is an analog of the results for $[-1, 1]$ by Teljakovskiĭ [17], Gopengauz [12] and DeVore [5,6]; see also Gonska and Hinnemann [10,13] Gonska et al. [11].

Note that in Theorem 2 one can replace lr by

$$\max\left\{r,\ \sum_{j:\alpha_j>1}\left(r-\left[\frac{r}{\alpha_j}\right]\right)\right\},$$

where $[a]$ stands for the integral part of a, but it is impossible to replace this by any smaller number, because polynomials, P_n and some of their derivatives must have prescribed values at all z_j's, for which $\alpha_j > 1$.

We wish to have inverse theorems as well. Although it is impossible to have a strong inversion, we have a weak one, with "extra" $\varepsilon > 0$. For Theorem 1 we have the following inverse.

Theorem 3. *Let $r \in \mathbb{N}$, $\varepsilon > 0$ and $\alpha_j \geqslant 1$ for all $j = 1, \ldots, l$. If a function f is analytic in G, then*

$$\left\|\varphi^r f^{(r)}\right\|_G \leqslant \frac{c}{\varepsilon} \sup_{n\geqslant r} n^{r+\varepsilon} E_n(f, G). \tag{1.2}$$

Anyway, if at least one $\alpha_j < 1$, then even this theorem fails to hold. Counterexamples, say $f(z) = z^r$, $f(z) = e^z$, etc. To salvage the idea, we have to seek a suitable additional condition, and we readily find it, because one can strengthen Theorem 1. Namely, we have

Theorem 4. *Let $r \in \mathbb{N}$. If a function f is analytic in G, then for each $n \geqslant r$ there is a polynomial $P_n \in \mathbb{P}_n$, such that*

$$\|f - P_n\|_G \leqslant \frac{c}{n^r}\left\|f^{(r)}\varphi^r\right\|_G, \tag{1.3}$$

and

$$\left\|P_n^{(r)}\varphi^r\right\|_G \leqslant c\left\|f^{(r)}\varphi^r\right\|_G. \tag{1.4}$$

So, if we agree to have (1.4) as an additional condition, then we obtain an inverse theorem without restrictions on $\alpha_j \in (0, 2]$. Moreover, we avoid the "extra" $\varepsilon > 0$, and we prove this statement, applying the integral Cauchy formulae for a circle only, and nothing more. Thus, we formulate

Theorem 5. *Let $r \in \mathbb{N}$. If a function f is analytic in G, then for each sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \mathbb{P}_n$ we have*

$$\left\|f^{(r)}\varphi^r\right\|_G \leqslant \liminf_{n\to\infty}\left(r!n^r\|f - P_n\|_G + \left\|P_n^{(r)}\varphi^r\right\|_G\right).$$

In contrast to the notation "constructive characterization", a pair of direct and inverse theorems, involving additional conditions, is called "approximative characterization"; see

[9, p. 267]. For the first results on approximative characterization see Zamanski [19] and Trigub [18]. Thus, Theorems 4 and 5 provide the approximative characterization of the class of analytic in G functions f, satisfying $\left\| f^{(r)}\varphi^r \right\|_G < +\infty$. We end the discussion on Theorem 1 with a corollary from Theorems 4 and 5.

Corollary 1. *Let $r \in \mathbb{N}$. If a function f is analytic in G, then there is a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \mathbb{P}_n$, such that*

$$\exists \lim_{n\to\infty} \left(r!\, n^r \, \|f - P_n\|_G + \left\| P_n^{(r)}\varphi^r \right\|_G \right) = \theta \left\| f^{(r)}\varphi^r \right\|_G ,$$

where $1 \leqslant \theta \leqslant c$.

Now we go back to the direct Theorem 2. Here we do not have problems with j's, for which $\alpha_j \leqslant 1$, but recall, we have problems with j's, for which $\alpha_j > 1$. This is a reason for the additional term $E_r(f, G)$ in inverse

Theorem 6. *Let $r \in \mathbb{N}$ and $\varepsilon > 0$. If a function f is analytic in G, then for each sequence $\{P_n\}_{n=lr}^{\infty}$ of polynomials $P_n \in \mathbb{P}_n$ we have*

$$\left\| f^{(r)} \right\|_G \leqslant c E_r(f, G) + \frac{c}{\varepsilon} \sup_{n \geqslant lr} n^{r+\varepsilon} \left\| \frac{f - P_n}{\varphi^r} \right\|_G . \tag{1.5}$$

In fact, we prove two more general theorems. Put

$$\alpha := \min\{1, \alpha_1, \ldots, \alpha_l\} . \tag{1.6}$$

Theorem 7. *Let $r \in \mathbb{N}$, and $0 \leqslant \beta \leqslant r$. If a function f is analytic in G, then for each $n \geqslant lr/\alpha$ there is a polynomial $P_n \in \mathbb{P}_n$, such that*

$$n^r \left\| \frac{(f - P_n)\,\varphi^\beta}{\varphi^r} \right\|_G + \left\| P_n^{(r)}\varphi^\beta \right\|_G \leqslant c \left\| f^{(r)}\varphi^\beta \right\|_G . \tag{1.7}$$

Recall, for $[-1, 1]$ a corresponding "*β-bridge*" is proved by Ditzian and Jiang [7].

Note that, if we delete $\|P_n^{(r)}\varphi^\beta\|_G$ in (1.7), then Theorem 7 is valid with lr, instead of lr/α, see Theorem in the Abstract to the paper.

Theorem 8. *Let $r \in \mathbb{N}$, and $0 \leqslant \beta \leqslant r$. If a function f is analytic in G, then for each sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \mathbb{P}_n$ we have*

$$\left\| f^{(r)}\varphi^\beta \right\|_G \leqslant \lim_{n\to\infty} \inf \left(r!\, n^r \left\| \frac{(f - P_n)\,\varphi^\beta}{\varphi^r} \right\|_G + \left\| P_n^{(r)}\varphi^r \right\|_G \right) . \tag{1.8}$$

Theorems 7 and 8 readily imply

Corollary 2. *Let $r \in \mathbb{N}$, and $0 \leqslant \beta \leqslant r$. If a function f is analytic in G, then there is a sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \mathbb{P}_n$, such that*

$$\exists \lim_{n\to\infty} \left(r!\, n^r \left\| \frac{(f - P_n)\, \varphi^\beta}{\varphi^r} \right\|_G + \left\| P_n^{(r)} \varphi^\beta \right\|_G \right) = \theta \left\| f^{(r)} \varphi^\beta \right\|_G ,$$

where $1 \leqslant \theta \leqslant c$.

Note that, Theorem 8 holds for each open set $\widetilde{G}$ and any continuous and positive in $\widetilde{G}$ function $\widetilde{\varphi}$ instead of G and φ, respectively. So, an interesting problem is to describe the set of all domains $\widetilde{G}$, for which there exists a continuous and positive in $\widetilde{G}$ function $\widetilde{\varphi}$, such that Theorem 7 holds.

We wish to emphasize that all Theorems 1–8 are the corollaries of Dzjadyk theory of approximation of functions on sets of complex plane. They are close to the results on this theory by Alibekov, Andrievskii, Bardzinskii, Belyi, Djuzkenkova, Dynkin, Galan, Lebedev, Polyakov, Shirokov, Shvay, Stovbun, Tamrazov, Volkov, Vorobyov and others; see their papers and [9,16,3,15].

In Section 2, we recall some results from the Dzjadyk theory. In Section 3, we prove some auxiliary lemmas. In Section 4, we prove an auxiliary theorem on simultaneous approximation, which readily implies Theorem 1. In Section 5, we prove Theorem 7, and hence Theorems 2 and 4. In Section 6 we prove inverse theorems.

Below we will have constants C that may depend on parameters other than G and r. We will indicate all these parameters in parentheses. In particular, $C(G, r) = c$. Constants c and C may differ even if they occur in the same line.

2. Dzjadyk polynomial kernels and Dzjadyk inequality

In this section in a form suitable for us we give some notations and results belonging to Dzjadyk, and in some important cases, to Lebedev, Tamrazov, Shirokov and Shevchuk. See [9,16,3,15] for the details.

Definition 1. For each point $z \in \mathbb{C}$ we denote by $j(z)$ the index of the closest to z among angle points $z_j,\ j = 1, \ldots, l$. If there are a few such closest angle points, then for the definiteness we denote by $j(z)$ the lowest index among them.

Note that under this definition we have

$$\varphi(z) \leqslant c \left| z - z_{j(z)} \right|^{1 - \frac{1}{\alpha_{j(z)}}} \leqslant c\varphi(z), \quad z \in \mathbb{C}, \quad z \neq z_{j(z)}. \tag{2.1}$$

Definition 2. For $n \in \mathbb{N}$, and $z \in \mathbb{C}$ set

$$\rho_n(z) := \begin{cases} n^{-\alpha_{j(z)}} & \text{if } \left|z - z_{j(z)}\right| \leqslant n^{-\alpha_{j(z)}}, \\ \frac{1}{n}\varphi(z) & \text{otherwise.} \end{cases} \tag{2.2}$$

Lemma D. *For all $n \in \mathbb{N}$, $z \in \overline{G}$ and $\zeta \in \overline{G}$ we have*

$$\rho_n^2(z) \leqslant c\left(|\zeta - z| + \rho_n(\zeta)\right)^{2-\alpha} \rho_n^{\alpha}(\zeta), \tag{2.3}$$

where α is defined by (1.6), *whence*

$$|\zeta - z| + \rho_n(z) \leqslant c\left(|\zeta - z| + \rho_n(\zeta)\right) \leqslant c(|\zeta - z| + \rho_n(z)). \tag{2.4}$$

Theorem D. *For each fixed $m \in \mathbb{N}$ and any $n \in \mathbb{N}$ there exists the Dzjadyk polynomial kernel $D_n(\zeta, z)$, $\zeta \in \partial G$, $z \in \mathbb{C}$, having the properties*

(a)

$$D_n(\zeta, z) = \sum_{k=0}^{n-1} a_k(\zeta) z^k,$$

where a_k, $k = 0, \ldots, n-1$, are continuous on ∂G functions;

(b) *for each $p = 0, 1, \ldots, m$, $\zeta \in \partial G$ and $z \in \overline{G}$, we have*

$$\left|\frac{\partial^p}{\partial z^p}\left(\frac{1}{\zeta - z} - D_n(\zeta, z)\right)\right| \leqslant \frac{C(G, m)}{|\zeta - z|^{p+1}} \left(\frac{\rho_n(z)}{|\zeta - z| + \rho_n(z)}\right)^m, \tag{2.5}$$

where $\zeta \neq z$, and

$$\left|\frac{\partial^p}{\partial z^p} D_n(\zeta, z)\right| \leqslant \frac{C(G, m)}{\left(|\zeta - z| + \rho_n(z)\right)^{p+1}}; \tag{2.6}$$

(c)

$$\frac{1}{2\pi i} \int_{\partial G} D_n(\zeta, z) d\zeta = 1; \text{ and} \tag{2.7}$$

(d) *if $P_m \in \mathbb{P}_m$, then, for all $z \in \overline{G}$ and $p = 0, 1, 2, \ldots$,*

$$\left|P_m^{(p)}(z) - \frac{1}{2\pi i} \int_{\partial G} P_m(\zeta) \frac{\partial^p}{\partial z^p} D_n(\zeta, z) d\zeta\right| \leqslant \frac{C}{n^m} \max_{j \geqslant p} |P_m^{(j)}(z)|, \tag{2.8}$$

where $C = C(G, m)$.

Inequality D. *Let $\gamma \in \mathbb{R}$. For each polynomial $P_n \in \mathbb{P}_n$ the Dzjadyk inequality*

$$\left\| P_n' \rho_n^{\gamma+1} \right\|_{\partial G} \leqslant C(\gamma, G) \left\| P_n \rho_n^{\gamma} \right\|_{\partial G} \tag{2.9}$$

holds, where $\|g\|_{\partial G} := \sup\{|g(z)| : z \in \partial G\}$.

3. Auxiliary lemmas

In contrast to Section 2, in Section 3 we do not use assumption (1.1), but everywhere we use assumption $\alpha \neq 0$. We begin with

Lemma 1. *For each $m \in \mathbb{N}$, $\sigma > -m$, $x_0 > 0$ and $x \geqslant 0$ the estimate*

$$\left|\int_{x_0}^{x} (x-u)^{m-1} u^{\sigma}\, du\right| \leqslant C\,|x-x_0|^m\, x_0^{\sigma} \left(1+\frac{|x-x_0|}{x_0}\right)^{\max\{0,\sigma\}} =: C J(\sigma, m, x_0, x) \tag{3.1}$$

holds, where $C = \max\left\{\frac{1}{m}, \frac{1}{m+\sigma}\right\}$. Moreover, if in addition $\sigma_1 > -m$ and $x_0 < x \leqslant a$, then

$$0 < \int_{x_0}^{x} (x-u)^{m-1} u^{\sigma} (a-u)^{\sigma_1}\, du \leqslant C J(\sigma, m, x_0, x), \tag{3.2}$$

where $C = \frac{1}{m+\sigma_1}\left(\frac{a-x_0}{2}\right)^{\sigma_1}$ if $\sigma_1 < 0$, and $C = \frac{1}{m} a^{\sigma_1}$ if $\sigma_1 \geqslant 0$.

Proof. If $\sigma < 0$ and $x < x_0$, then

$$\begin{aligned} \mu &:= \left|\int_{x_0}^{x} (x-u)^{m-1} u^{\sigma}\, du\right| = \int_{x}^{x_0} (u-x)^{m-1+\sigma} \left(1-\frac{x}{u}\right)^{-\sigma} du \\ &\leqslant \left(1-\frac{x}{x_0}\right)^{-\sigma} \int_{x}^{x_0} (u-x)^{m-1+\sigma}\, du = \frac{|x-x_0|^m x_0^{\sigma}}{m+\sigma}. \end{aligned}$$

If either $\sigma \geqslant 0$ and $x \leqslant x_0$, or $\sigma < 0$ and $x \geqslant x_0$, then $\mu \leqslant \frac{1}{m}\, |x-x_0|^m\, x_0^{\sigma}$. Finally, if $\sigma \geqslant 0$ and $x > x_0$, then

$$\mu \leqslant \frac{1}{m}\,(x-x_0)^m x^{\sigma} = \frac{1}{m}\,(x-x_0)^m x_0^{\sigma} \left(1+\frac{x-x_0}{x_0}\right)^{\sigma}.$$

So (3.1) is proved. Now we verify (3.2). If $\sigma_1 < 0$ and $x > \frac{x_0+a}{2}$, then $x - x_0 > \frac{a-x_0}{2}$, whence

$$\begin{aligned} \nu &:= \int_{x_0}^{x} (x-u)^{m-1} u^{\sigma} (a-u)^{\sigma_1}\, du \\ &\leqslant \max\{x_0^{\sigma}, x^{\sigma}\} \int_{x_0}^{x} (x-u)^{m-1+\sigma_1}\, du \\ &= \frac{1}{m+\sigma_1}\, \max\left\{x_0^{\sigma}, x^{\sigma}\right\} (x-x_0)^m (x-x_0)^{\sigma_1} \\ &\leqslant \frac{1}{m+\sigma_1} \left(\frac{a-x_0}{2}\right)^{\sigma_1} J(\sigma, m, x_0, x). \end{aligned}$$

If $\sigma_1 < 0$ and $x \leqslant \frac{x_0+a}{2}$, then $a - u \geqslant \frac{a-x_0}{2}$, whence $\nu \leqslant \left(\frac{a-x_0}{2}\right)^{\sigma_1} \mu$. Finally, if $\sigma_1 \geqslant 0$, then $\nu \leqslant a^{\sigma_1} \mu$. $\square$

We need two definitions.

Definition 3. Let $\gamma(z_*, z^*)$ be a simple Jordan-rectifiable curve with the endpoints z_* and z^*, and let $\zeta = \zeta(s)$, $s_* \leqslant s \leqslant s^*$, be its natural parametrization, that is, $s - s_*$ is the length of the arc $\gamma(z_*, \zeta)$ of the curve $\gamma(z_*, z^*)$, with the endpoints $z_* = \zeta(s_*)$ and $\zeta = \zeta(s)$. We will write

$$\gamma(z_*, z^*) \subset L(\lambda),$$

where $\lambda = const > 0$, if for each $s' \in [s_*, s^*]$ and $s \in [s_*, s^*]$ the inequality

$$|s' - s| \leqslant \lambda |\zeta(s') - \zeta(s)|$$

holds, that is

$$|\zeta(s') - \zeta(s)| \leqslant |s' - s| \leqslant \lambda |\zeta(s') - \zeta(s)|. \tag{3.3}$$

Note that, some authors call such a curve as a “quasismooth curve”.

Definition 4. We will say that a curve $\gamma(z_*, z^*)$ is a proper curve, if it is a simple Jordan curve with the endpoints $z_* \in \overline{G}$ and $z^* \in \overline{G}$, and $\gamma(z_*, z^*) \setminus \{z_*, z^*\} \subset G$.

Since $\alpha_j \neq 0$, $j = 1, \ldots, l$, then the following Lemmas 2 and 3 are more or less obvious. To prove them one can use, say, the arguments in [9, Chapter IX.4]. If in addition all $\alpha_j \neq 2$, then Lemma 3 follows, say, from Lemma 2.2 in [1].

Lemma 2. *Every two points $z_* \in \overline{G}$ and $z^* \in \overline{G}$ can be connected by a proper curve $\gamma(z_*, z^*) \in L(c)$.*

Lemma 3. *Let $z_0 \in G$ and $z^0 \in G$, $j(z_0) =: j_0$, $j(z^0) =: j^0$. Then* (a) *if $j_0 \neq j^0$, then there is a proper curve $\gamma := \gamma(z_{j_0}, z_{j^0}) \subset L(c)$, such that $z_0 \in \gamma$, $z^0 \in \gamma$, and for all $j = 1, \ldots, l$, $j \neq j_0$, $j \neq j^0$, we have*

$$\left|z - z_j\right| \geqslant c, \quad z \in \gamma; \tag{3.4}$$

(b) *if $j_0 = j^0$ then there is a proper curve $\gamma := \gamma(z_{j_0}, \widetilde{z}) \subset L(c)$, where either $\widetilde{z} = z_0$, or $\widetilde{z} = z^0$, such that $z_0 \in \gamma$, $z^0 \in \gamma$, and for all $j = 1, \ldots, l$, $j \neq j_0$, we have*

$$\left|z - z_j\right| \geqslant c, \quad z \in \gamma. \tag{3.5}$$

Everywhere below $r \in \mathbb{N}$, $0 \leqslant \beta \leqslant r$, and a *function f is analytic in G*. Let $T(z_0, z)$ be the $r - 1$ -st Taylor polynomial

$$T(z_0, z) := f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^{(r-1)}(z_0)}{(r-1)!}(z - z_0)^{r-1}$$

of the function f at the point $z_0 \in G$.

Lemma 4. *If* $\left\| f^{(r)}\varphi^{\beta} \right\|_G = 1$, *then for all* $p = 0, 1, \ldots, r-1-[\frac{\beta}{2}]$, $z_0 \in G$ *and* $z \in G$ *the estimate*

$$\left| f^{(p)}(z) - T^{(p)}(z_0, z) \right| \leqslant C\, \frac{|z-z_0|^{r-p}}{\varphi^{\beta}(z_0)} \left(1 + \frac{|z-z_0|}{|z_0 - z_{j(z_0)}|} \right)^{\frac{r}{\alpha}} \tag{3.6}$$

holds, where $C = c(2(r-p)-\beta)^{-1}$, *and, recall, c may depend only on G and r.*

Proof. We represent $f^{(p)} - T^{(p)}$ in the form

$$\begin{aligned} f^{(p)}(z) - T^{(p)}(z_0, z) &= \frac{1}{(r-p-1)!} \int_{z_0}^{z} (z-\zeta)^{r-p-1} f^{(r)}(\zeta)\, d\zeta \\ &=: \frac{1}{(r-p-1)!}\tau(p, z, z_0). \end{aligned}$$

So to prove (3.6) we have to estimate $|\tau(p, z, z_0)|$. Assume that $j(z_0) = 1$, and consider two cases. First let $j(z) = 1$ as well. Then we denote a proper curve by γ, guaranteed by (b) of Lemma 3 for the points z_0 and $z^0 := z$. Let $\zeta(s)$ be its natural parametrization, $z_1 = \zeta(0)$, $z_0 = \zeta(s_0)$, and $z = \zeta(s^0)$, and γ_0 be the arc of γ with the endpoints z_0 and z. By (3.5) and (2.1),

$$\varphi(\zeta) \leqslant c\, |\zeta - z_1|^{1-\frac{1}{\alpha_1}} \leqslant c\varphi(\zeta), \quad \zeta \in \gamma \setminus \{z_1\}, \tag{3.7}$$

whence

$$\begin{aligned} |\tau(p, z, z_0)| &\leqslant c \int_{\gamma_0} |z-\zeta|^{r-p-1}\, |\zeta - z_1|^{\frac{\beta}{\alpha_1}-\beta}\, |d\zeta| \\ &\leqslant c \left| \int_{s_0}^{s^0} (s^0 - s)^{r-p-1} s^{\frac{\beta}{\alpha_1}-\beta}\, ds \right|. \end{aligned}$$

Since $r - p \geqslant \left[\frac{\beta}{2}\right] + 1 > \frac{\beta}{2}$ by the condition of Lemma 4, then

$$r - p + \frac{\beta}{\alpha_1} - \beta \geqslant r - p - \frac{\beta}{2} > 0.$$

Therefore we may apply (3.1) of Lemma 1 and obtain

$$|\tau(p, z, z_0)| \leqslant \frac{c}{2(r-p)-\beta}\, J\left(\frac{\beta}{\alpha_1} - \beta,\ r-p,\ s_0,\ s^0 \right).$$

This and (3.7) imply (3.6) in the case $j(z_0) = j(z)$. Now let $j(z) =: j^0 \neq 1$. Then we denote by γ a proper curve, guaranteed by (a) of Lemma 3 for the points z_0 and $z^0 := z$. Let $\zeta(s)$ be its natural parametrization, $z_1 = \zeta(0)$, $z_0 = \zeta(s_0)$, $z = \zeta(s^0)$, and $z_{j^0} = \zeta(a)$. By (3.4) and (2.1),

$$\varphi(\zeta) \leqslant c\, |\zeta - z_1|^{1-\frac{1}{\alpha_1}} \left| \zeta - z_{j^0} \right|^{1-\frac{1}{\alpha_{j^0}}} \leqslant c\varphi(\zeta), \quad \zeta \in \gamma \setminus \{z_1, z_{j^0}\}.$$

First we assume that $s_0 \leqslant s^0$. Then we repeat the arguments of the previous case with (3.2) instead of (3.1), and obtain (3.6). One should only notice that

$$a \leqslant c\,(diam\ G) = c, \quad \text{and} \quad a - s_0 \geqslant c\left|z_{j^0} - z_0\right| \geqslant c\left|z_{j^0} - z_1\right| = c.$$

Otherwise, if $s^0 < s_0$, then, for all $s \in [s^0, s_0]$,

$$\frac{\left|z_0 - z_{j^0}\right|}{\left|\zeta(s) - z_{j^0}\right|} \leqslant c\,\frac{a - s_0}{a - s} \leqslant c, \ \text{and} \ \frac{|z - z_1|}{|\zeta(s) - z_1|} \leqslant c\,\frac{s^0}{s} \leqslant c,$$

that is $\left|\zeta(s) - z_{j^0}\right| > c$ and $|\zeta(s) - z_1| > c$, $s \in [s^0, s_0]$. Therefore $c < \varphi(\zeta(s)) < c$ for $s \in [s^0, s_0]$. Hence

$$|\tau(p, z, z_0)| \leqslant c\,|z - z_0|^{r-p} \leqslant c\,|z - z_0|^{r-p}\,\varphi^{-\beta}(z_0).$$

The lemma is proved. □

Let $\left\|f^{(r)}\varphi^{\beta}\right\|_G < +\infty$. If either $r > 2$, or $r = 2 > \beta$, then the case $p = 1$ of Lemma 4 implies $\|f'\|_G < \infty$. The same holds, if $r = \beta = 2 > \alpha_j, \quad j = 1, \ldots, l$. Otherwise, that is, if either $r = 1$, or $r = \beta = 2 = \alpha_j$ for some j, then evidently

$$\left|f'(z)\right| \leqslant \frac{C}{\sqrt{|z - z_1| \cdots |z - z_l|}}, \quad z \in G,$$

where C does not depend on z. Hence f can be continuously extended on the closure $\overline{G}$ of G by

$$f(z) = f(z_0) + \int_{z_0}^{z} f'(\zeta)\,d\zeta,$$

where $z_0 \in G$ is a fixed point.

So, everywhere below without loss of generality we assume that a *function f is continuous on $\overline{G}$, if* $\left\|f^{(r)}\varphi^{\beta}\right\|_G < +\infty$.

Lemma 4, Definition 2 of ρ_n, and the estimate $\left|z_{j(z_0)} - z_0\right| \geqslant c\rho_n(z_0)$ for $\left|z_{j(z_0)} - z_0\right| \geqslant n^{-\alpha_{j_0}}$ readily imply

Lemma 5. *Let $n \in \mathbb{N}$, $z_0 \in G$, $j(z_0) =: j_0$, and $\left\|\varphi^{\beta} f^{(r)}\right\|_G = 1$. We have* (a) *if $\left|z_0 - z_{j_0}\right| \geqslant n^{-\alpha_{j_0}}$, then*

$$|f(z) - T(z_0, z)| \leqslant \frac{c}{n^{\beta}}\,\frac{|z - z_0|^r}{\rho_n^{\beta}(z_0)}\left(1 + \frac{|z - z_0|}{\rho_n(z_0)}\right)^{\frac{r}{\alpha}}, \quad z \in \overline{G}, \tag{3.8}$$

(b) *if $z \in G$ and $\left|z - z_{j_0}\right| \leqslant \left|z_0 - z_{j_0}\right| = n^{-\alpha_{j_0}}$, then, for all $p = 0, \ldots, r - 1 - [\frac{\beta}{2}]$,*

$$\left|f^{(p)}(z) - T^{(p)}(z_0, z)\right| \leqslant \frac{1}{2(r-p) - \beta}\,\frac{c}{n^{\beta}}\,\rho_n^{r-p-\beta}(z_0) \tag{3.9}$$

holds, where evidently

$$\frac{1}{2(r-p)-\beta} < 1, \tag{3.10}$$

if either $p = r - [\beta]$ *and* $\beta \geqslant 2$, *or* $p \leqslant r - 1 - [\beta]$.

We end the Section with Lemma 6. Denote by $\hat{z} \in G$ a point, such that $c \leqslant \varphi(\hat{z}) \leqslant C(G)$. Let say $\hat{z}$ be the center of the largest open disk, inscribed in G, or one of them.

Lemma 6. *Let* $\left\| f^{(r)} \varphi^{\beta} \right\|_G = 1$. *If* $f^{(p)}(\widehat{z}) = 0$ *for all* $p = 0, \ldots, r-1$; *then for all these* p *we have*

$$\left| f^{(p)}(z) \right| \leqslant \frac{c}{\left| z - z_{j(z)} \right|^r}, \quad z \in G. \tag{3.11}$$

Proof. Let, say $j(z) = 1$, and $j(\hat{z}) =: \hat{\jmath} \neq 1$. Then we denote by $\gamma = \gamma(z_1, z_{\hat{\jmath}})$ a proper curve, guaranteed by (a) of Lemma 3 for the points $z_0 := \hat{z}$ and $z^0 := z$. Let $\zeta(s)$ be its natural parametrization, $z_1 = \zeta(0)$, $z = \zeta(s^0)$ and $\hat{z} = \zeta(\hat{s})$. Note that for all $\zeta = \zeta(s)$, such that $0 < s \leqslant \hat{s}$,

$$|f^{(r)}(\zeta)| \leqslant \varphi^{-\beta}(\zeta) \leqslant c|\zeta - z_1|^{\beta(\frac{1}{\alpha_1}-1)} \leqslant cs^{\beta(\frac{1}{\alpha_1}-1)} \leqslant cs^{-r/2}. \tag{3.12}$$

Now, if $s^0 \leqslant \hat{s}$, then

$$\begin{aligned} \left| f^{(p)}(z) \right| &= \frac{1}{(r-p-1)!} \left| \int_z^{\hat{z}} (z-\zeta)^{r-p-1} f^{(r)}(\zeta)\, d\zeta \right| \\ &\leqslant c \int_{s^0}^{\hat{s}} \frac{1}{s^{r/2}}\, ds \leqslant \frac{c}{(s^0)^{r/2}} \leqslant \frac{c}{(s^0)^r} \leqslant \frac{c}{\left| z - z_{j(z)} \right|^r}. \end{aligned}$$

Otherwise, if $s^0 > \hat{s}$, then $\left| f^{(p)}(z) \right| \leqslant c$. The lemma is proved. □

4. Proof of Theorem 1

Since in all direct theorems we suppose that $n \geqslant r$ at least, then without loss of generality we may assume that at $\hat{z} \in D$ we have

$$f^{(p)}(\hat{z}) = 0 \tag{4.1}$$

for all $p = 0, \ldots, r-1$. Recall that we defined $\hat{z}$ as the center of the largest open disk, inscribed in G or one of them.

Everywhere below

$$r^* := \begin{cases} \frac{r}{\alpha} & \text{if } \frac{r}{\alpha} \text{ is integer,} \\ 1 + \left[\frac{r}{\alpha}\right] & \text{otherwise.} \end{cases} \tag{4.2}$$

Now we prove an auxiliary Theorem 9, and Theorem 1 is a particular case of this theorem. Recall, we assumed, that a function f is analytic in G, and continuous on $\overline{G}$ if $\left\|\varphi^\beta f^{(r)}\right\|_G < +\infty$.

Theorem 9. *Let* $r \in \mathbb{N}$, $0 \leqslant \beta \leqslant r$, $\left\|\varphi^\beta f^{(r)}\right\|_G = 1$, $D_n(\zeta, z)$ *is the Dzjadyk polynomial kernel, defined by Theorem D for* $m = 5r^*$, *and*

$$P_n(z) = \frac{1}{2\pi i} \int_{\partial G} f(\zeta) D_n(\zeta, z)\, d\zeta, \tag{4.3}$$

the polynomial of degree $< n$. *If* (4.1) *holds, then for each* $p = 0, \ldots, r - [\frac{\beta}{2}] - 1$, *we have*

$$\left|f^{(p)}(z) - P_n^{(p)}(z)\right| \leqslant \frac{1}{2(r-p)-\beta} \frac{c}{n^\beta} \rho_n^{r-p-\beta}(z), \quad z \in G, \tag{4.4}$$

and, for all $p = r, \ldots, r^*$,

$$\left|P_n^{(p)}(z)\right| \leqslant \frac{c}{n^\beta} \rho_n^{r-p-\beta}(z), \quad z \in G. \tag{4.5}$$

Proof. We follow the Dzjadyk scheme; see say [15, Lemma 21.2]. We fix $p \leqslant r^*$ and $z \in G$. To avoid too much writings we will write ρ instead of $\rho_n(z)$, that is everywhere below in the proof $\rho = \rho_n(z)$. Now we denote a point z_0. If $\left|z - z_{j(z)}\right| > n^{-\alpha_{j(z)}}$, then we put $z_0 := z$. Otherwise we denote by $z_0 \in G$ any fixed point, such that

$$\left|z_0 - z_{j(z)}\right| = n^{-\alpha_{j(z)}}.$$

By Definition 2, $\rho_n(z_0) = \rho$ as well. We put

$$g(\zeta) := f(\zeta) - T(z_0, \zeta),$$

and note that $f^{(p)}(z) = T^{(p)}(z_0, z)$, if $z = z_0$. Let U be the closed disk with the center at z_0, and of the radius 2ρ, and ∂U – its boundary. Now let us represent $P_n^{(p)} - T^{(p)}$ in the form

$$\begin{aligned}
&2\pi i \left(P_n^{(p)}(z) - T^{(p)}(z_0, z)\right) \\
&\quad = \int_{\partial G \setminus U} g(\zeta) \frac{\partial^p}{\partial z^p} \left(D_n(\zeta, z) - \frac{1}{\zeta - z}\right) d\zeta \\
&\qquad + \int_{\partial G \cap U} g(\zeta) \frac{\partial^p}{\partial z^p} D_n(\zeta, z)\, d\zeta \\
&\qquad + p! \int_{\partial G \setminus U} g(\zeta) (\zeta - z)^{-p-1}\, d\zeta \\
&\qquad + \left(\int_{\partial G} T(z_0, \zeta) \frac{\partial^p}{\partial z^p} D_n(\zeta, z) d\zeta - 2\pi i T^{(p)}(z_0, z)\right) \\
&\quad =: i_1 + i_2 + i_3 + i_4.
\end{aligned}$$

Inequalities (3.8), (2.5) and (2.6) yield

$$|i_1| + |i_2| \leqslant c \int_{\partial G} \frac{(|\zeta - z| + \rho)^{r-p-1}}{n^\beta \rho^\beta} \left(\frac{\rho}{|\zeta - z| + \rho} \right)^{m - \frac{r}{\alpha}} |d\zeta|$$
$$\leqslant \frac{c}{n^\beta} \rho^{r-p-\beta} \int_{\partial G} \frac{\rho}{|\zeta - z|^2 + \rho^2} |d\zeta| \leqslant \frac{c}{n^\beta} \rho^{r-p-\beta},$$

where for the last estimate see, say [15, (21.15)]. Now, since the point z lies outside the domain (domains), bounded by $(\partial G \backslash U) \cup (\partial U \cap G)$, then

$$\frac{i_3}{p!} = \int_{(\partial G \backslash U) \cup (\partial U \cap G)} g(\zeta)\, (\zeta - z)^{-p-1}\, d\zeta - \int_{\partial U \cap G} g(\zeta)\, (\zeta - z)^{-p-1}\, d\zeta$$
$$= - \int_{\partial U \cap G} g(\zeta)\, (\zeta - z)^{-p-1}\, d\zeta.$$

For $\zeta \in \partial U$ we have $\rho \leqslant |\zeta - z|$; hence, (3.8) yields

$$|i_3| \leqslant \frac{c}{n^\beta} \rho^{r-\beta} \int_{\partial U \cap G} \rho^{-p-1} |d\zeta| \leqslant \frac{c}{n^\beta} \rho^{r-\beta-p-1} \int_{\partial U} |d\zeta| = \frac{c}{n^\beta} \rho^{r-p-\beta}.$$

Then, for all $\zeta \in G$ (3.11) and (4.1) imply

$$\max_{j \geqslant p} \left| T^{(j)}(\zeta, z_0) \right| \leqslant c \max_{j=0,\dots,r-1} \left| f^{(j)}(z_0) \right| \leqslant \frac{c}{\left| z_0 - z_{j(z_0)} \right|^r} \leqslant c n^{2r}.$$

So we apply (2.8) and obtain

$$|i_4| \leqslant c n^{2r-m} \leqslant \frac{c}{n^{3r^*}} \leqslant \frac{c}{n^\beta} \rho^{r-p-\beta},$$

where in the last inequality we used the estimate $\left\| \rho_n \right\|_G \geqslant \frac{c}{n^2}$.

Thus, for all $p = 0, \dots, r^*$ and $z \in G$ we have proved the inequality

$$\left| P_n^{(p)}(z) - T^{(p)}(z_0, z) \right| \leqslant \frac{c}{n^\beta} \rho^{r-p-\beta}. \tag{4.6}$$

Now (4.5) is evident, since $T^{(p)} \equiv 0$ for $p \geqslant r$. Estimate (4.4) is proved for all $z \in G$, satisfying $\left| z - z_{j(z)} \right| > n^{-\alpha_{j(z)}}$, since $T^{(p)}(z_0, z) = f^{(p)}(z)$ for these z (and, by the way, for all $p = 0, \dots, r-1$, not only for $p \leqslant r - 1 - [\frac{\beta}{2}]$). Finally, for $p \leqslant r - 1 - [\frac{\beta}{2}]$ and $z \in G$, satisfying $\left| z - z_{j(z)} \right| \leqslant n^{-\alpha_{j(z)}}$, estimate (4.4) follows from (3.9), (4.6) and the inequality

$$\left| f^{(p)} - P_n^{(p)} \right| \leqslant \left| f^{(p)} - T^{(p)} \right| + \left| T^{(p)} - P_n^{(p)} \right|.$$

Theorem 9 is proved. □

Note that, since $\left\| \rho_n \right\|_G \to 0, \quad n \to \infty, \quad \left\| \rho_n^{-1} \right\|_G < +\infty$, and $[\beta/2] + 1 \leqslant [\beta]$ for $\beta \geqslant 1$, then Theorem 9 implies

Corollary 3. *Let $\beta \neq r$. If $\left\| f^{(r)}\varphi^{\beta} \right\|_G < +\infty$, then the function f is $r-[\beta]-1$ times continuously differentiable on the closure $\overline{G}$ of G, and if $[\beta] \neq 0$, then*

$$\left\| f^{(r-[\beta])} \right\|_G < +\infty. \tag{4.7}$$

For $\beta = r$ we have

Proof of Theorem 1. Theorem 1 follows from the case $\beta = r$ and $p = 0$ of Theorem 9 and the inequality $2r - [\frac{r}{2}] \geqslant 2$. □

5. Proof of Theorem 7

We need two lemmas. Let r^* be defined by (4.2).

Lemma 7. *For each fixed $j^* = 1, \ldots, l$, $p = 0, \ldots, r^* - 1$ and any $n \geqslant lr^*$ there is a polynomial $Q_{j^*,p} \in \mathbb{P}_n$, satisfying* (a)

$$Q^{(p)}_{j^*,p}(z_{j^*}) = 1, \tag{5.1}$$

(b) *for all $j = 1, \ldots, l$ and $q = 0, \ldots, r^* - 1$, except $(j = j^*, q = p)$,*

$$Q^{(q)}_{j^*,p}(z_j) = 0, \tag{5.2}$$

holds, and (c) *for all $q = 0, \ldots, r^*$ we have*

$$\left| Q^{(q)}_{j^*,p}(z) \right| \leqslant \frac{c\rho_n^{r^*}(z_{j^*})\rho_n^{r^*}(z)}{\left(\left|z - z_{j^*}\right| + \rho_n(z)\right)^{2r^*-p+q}}, \quad z \in \overline{G}. \tag{5.3}$$

Proof. Without loss of generality assume that $j^* = 1$. If $n = lr^*$, then the polynomial $Q_{1,p}$, satisfying (5.1) and (5.2) is unique, and it satisfies (5.3) as well, since $c \leqslant \rho_{lr^*}(z) \leqslant c$, $z \in \overline{G}$. If $lr^* \leqslant n \leqslant 2lr^* + 1$, then again $c < \rho_n(z) < c$, $z \in \overline{G}$; therefore, one can take the same polynomial for these n. So below is the proof $n > 2lr^* + 1$. Then, for $m = 2[r^*/\alpha] + 2 + r^*$ and $\zeta = z_1$ we take the Dzjadyk polynomial kernel $D_{n_1}(z_1, z)$, given by Theorem D, where $n_1 = [n/2] - lr^*$. For each $p = 0, \ldots, r^* - 1$, denote by

$$Q_p(z) := \left(\prod_{j=2}^{l} \frac{z - z_j}{z_1 - z_j} \right)^{r^*} \frac{(z - z_1)^p}{p!} (1 - (z_1 - z)D_{n_1}(z_1, z)),$$

the polynomial of degree $< n/2$. Evidently,

$$Q_p^{(\nu)}(z_1) = 0, \quad \nu = 0, \ldots, p - 1,$$

$$Q_p^{(\nu)}(z_j) = 0, \quad \nu = 0, \ldots, r^* - 1, \quad j = 2, \ldots, l,$$

and

$$Q_p^{(p)}(z_1) = 1.$$

Now, for all $v = 0, \ldots, m$, we verify the estimate

$$\left| \frac{d^v}{dz^v} Q_p(z) \right| \leqslant \frac{c\rho^m}{(|z - z_1| + \rho)^{m-p+v}}, \quad z \in \overline{G}, \tag{5.4}$$

where we write ρ instead of $\rho_n(z)$ again. Indeed, if $|z - z_1| \geqslant \rho$ and $\mu \leqslant v$, then (2.5) implies

$$\begin{aligned} \kappa := \left| \frac{d^\mu}{dz^\mu} (z - z_1)^p (1 - (z_1 - z) D_{n_1}(z_1, z)) \right| &\leqslant \frac{c\rho^m}{|z - z_1|^{m-p+\mu}} \\ &\leqslant \frac{c\rho^m}{(|z - z_1| + \rho)^{m-p+v}}, \end{aligned}$$

where we used the inequalities $\rho \leqslant \rho_{n_1}(z) \leqslant c\rho$. If $|z - z_1| < \rho$ and $\mu \leqslant v$, then (2.6) yields

$$\kappa \leqslant \frac{c}{\rho^{\mu-p}} \leqslant \frac{c}{\rho^{v-p}} = \frac{c\rho^m}{\rho^{m+v-p}} \leqslant \frac{c\rho^m}{(|z - z_1| + \rho)^{m-p+v}}.$$

That is, (5.4) holds, since, for all $\mu = 0, 1, 2, \ldots$,

$$\left| \frac{d^\mu}{dz^\mu} \left(\prod_{j=2}^{l} \frac{z - z_j}{z_1 - z_j} \right)^{r^*} \right| \leqslant c, \quad z \in \overline{G}.$$

Since $\frac{\alpha}{2}(m - r^*) > r^*$, then Lemma D implies

$$\left(\frac{\rho}{|z - z_1| + \rho} \right)^{m-r^*} \leqslant c \left(\frac{\rho_n(z_1)}{|z - z_1| + \rho} \right)^{\frac{\alpha}{2}(m-r^*)} \leqslant c \left(\frac{\rho_n(z_1)}{|z - z_1| + \rho} \right)^{r^*}.$$

Hence (5.4) yields, for all $v = 0, \ldots, m$,

$$\left| \frac{d^v}{dz^v} Q_p(z) \right| \leqslant \frac{c\rho_n^{r^*}(z_1)\rho^{r^*}}{(|z - z_1| + \rho)^{2r^*-p+v}}, \quad z \in \overline{G}.$$

Finally, we put $Q_{1,r^*-1} := Q_{r^*-1}$, and, for $p = 0, \ldots, r^* - 2$,

$$Q_{1,p} := Q_p - \sum_{v=p+1}^{r^*-1} Q_p^{(v)} Q_{1,v},$$

and see that $Q_{1,p}$ is a required polynomial. The lemma is proved. □

Lemma 7 and (2.4) of Lemma D imply

Lemma 8. *Let the numbers $\sigma_{j,v}$ be given, such that*

$$|\sigma_{j,v}| \leqslant 1, \quad j = 1, \ldots, l, \quad v = 0, \ldots, r^* - 1.$$

Then the polynomial

$$Q_n(z) := \sum_{j=1}^{l} \sum_{v=0}^{r^*-1} \sigma_{j,v} \rho_n^{r-\beta-v}(z_j) Q_{j,v}(z) \tag{5.5}$$

of degree $< n$ satisfies, for all $q = 0, \dots, r^*$,

$$\left| Q_n^{(q)}(z) \right| \leqslant c \rho_n^{r-\beta-q}(z), \quad z \in \overline{G}, \tag{5.6}$$

and

$$Q_n^{(q)}(z_j) = \sigma_{j,q} \rho_n^{r-\beta-q}(z_j), \quad j = 1, \dots, l, \quad q \neq r^*. \tag{5.7}$$

Proof of Theorem 7. If $\left\| f^{(r)} \varphi^{\beta} \right\|_G = \infty$, then there is nothing to prove. So assume that $\left\| f^{(r)} \varphi^{\beta} \right\|_G = 1$. Then the function f is continuous on $\overline{G}$, and by Corollary 3 f has $r - [\beta] - 1$ continuous derivatives on $\overline{G}$. Denote by $R_n \in \mathbb{P}_n$ the polynomial, defined by the right-hand side of (4.3). Now, for all $j = 0, \dots, l$ and $v = 0, \dots, r^* - 1$ we define numbers $\sigma_{j,v}$. If $\alpha_j > 1$, then we put

$$\sigma_{j,v} := \begin{cases} \frac{f^{(v)}(z_j) - R_n^{(v)}(z_j)}{n^{-\beta} \rho_n^{r-\beta-v}(z_j)} & \text{if } v < r - [\beta], \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

If $\alpha_j < 1$, then we put

$$\sigma_{j,v} := \begin{cases} 0 & \text{if } v < r, \\ \frac{-R_n^{(v)}(z_j)}{n^{-\beta} \rho_n^{r-v-\beta}(z_j)} & \text{otherwise.} \end{cases} \tag{5.9}$$

If $\alpha_j = 1$, then we put $\sigma_{j,v} = 0$ for all j and v. Theorem 9 and (3.10) yield

$$\left| \sigma_{j,v} \right| \leqslant c, \quad j = 0, \dots, l, \quad v = 0, \dots, r^* - 1.$$

Let $Q_n \in \mathbb{P}_n$ be the polynomial, defined by (5.5). Below we prove that the polynomial

$$P_n := R_n + \frac{1}{n^{\beta}} Q_n$$

is required in Theorem 7. That is,

$$|f(z) - P_n(z)| \leqslant \frac{c}{n^r} \varphi^{r-\beta}(z), \quad z \in G, \tag{5.10}$$

and

$$\left| P_n^{(r)}(z) \right| \leqslant c \varphi^{-\beta}(z), \quad z \in G. \tag{5.11}$$

To this end we set

$$G_j := \left\{ z \in G : \left| z - z_j \right| \leqslant \rho_n(z_j) \right\}.$$

Now, if $z \in G \setminus \bigcup_{j=1}^{l} G_j$, then by its definitions, $\rho_n(z) = \frac{1}{n}\varphi(z)$; hence Theorem 9 and (5.6) imply that for such z both estimates (5.10) and (5.11) hold. If $\alpha_j > 1$, then $\rho_n(z) \geqslant \frac{c}{n}\varphi(z)$, $z \in G_j$; hence, for such z estimate (5.11) follows from (5.6) and (4.5) for $q = r$ and $p = r$ respectively. If $\alpha_j < 1$, then (5.7) and (5.9) yield $P_n^{(v)}(z_j) = 0$ for all $v = r, \ldots, r^* - 1$; therefore, for $z \in G_j$, we have

$$P_n^{(r)}(z) = \frac{1}{(r^* - r - 1)!} \int_{z_j}^{z} (z - \zeta)^{r^* - r - 1} P_n^{(r^*)}(\zeta)\, d\zeta,$$

whence (5.6) for $q = r^*$, (4.5) for $p = r^*$, and (2.1) imply

$$\begin{aligned} \left| P_n^{(r)}(z) \right| &\leqslant c \left| z - z_j \right|^{r^* - r} \frac{1}{n^\beta} \left(\frac{1}{n} \right)^{\alpha_j (r - r^* - \beta)} \\ &\leqslant c \left| z - z_j \right|^{-\beta(1 - \frac{1}{\alpha_j})} \leqslant c\varphi^{-\beta}(z). \end{aligned}$$

So, (5.11) is proved. Thus we have to prove (5.10) for $z \in \bigcup_{j=1}^{l} G_j$. If $\alpha_j < 1$ then $\rho_n(z) \leqslant \frac{c}{n}\varphi(z)$, $z \in G_j$; hence (5.10) follows from (5.6) for $q = 0$ and (4.4) for $p = 0$. Now, let $z \in G_j$, with $\alpha_j > 1$ and $\beta \neq r$ (if $\beta = r$, then (5.10) readily follows from (4.4) and (5.6)). By (5.8) and (5.7),

$$f^{(q)}(z_j) - P_n^{(q)}(z_j) = 0, \quad q = 0, \ldots, r - [\beta] - 1;$$

therefore

$$f(z) - P_n(z) = \int_{z_j}^{z} \frac{(z - \zeta)^{r - [\beta] - 1}}{(r - [\beta] - 1)!} \left(f^{(r - [\beta])}(\zeta) - P_n^{(r - [\beta])}(\zeta) \right) d\zeta, \tag{5.12}$$

where the integral is well defined. For $[\beta] \neq 0$ this is guaranteed by (4.7). Now, for $\beta \geqslant 2$, (5.10) follows from (5.6), (4.4) and (3.10).

If $\beta < 1$, and recall $\alpha_j > 1$ and $\left| z - z_j \right| \leqslant n^{-\alpha_j}$, then we take a proper curve $\gamma(z_j, z) \in L(c)$, and obtain

$$\begin{aligned} \left| \int_{z_j}^{z} (z - \zeta)^{r-1} f^{(r)}(\zeta)\, d\zeta \right| &\leqslant c \int_{\gamma(z_j, z)} |z - \zeta|^{r-1} \left| \zeta - z_j \right|^{\frac{\beta}{\alpha_j} - \beta} |d\zeta| \\ &\leqslant c \left| z - z_j \right|^{r + \frac{\beta}{\alpha_j} - \beta} \\ &= c \left| z - z_j \right|^{(r - \beta)(1 - \frac{1}{\alpha_j})} \left| z - z_j \right|^{\frac{r}{\alpha_j}} \\ &\leqslant \frac{c}{n^r} \varphi^{r - \beta}(z), \end{aligned}$$

where we used (2.1). Therefore (5.6) and (4.5) imply

$$\begin{aligned} |f(z) - P_n(z)| &= \frac{1}{(r-1)!} \left| \int_{z_j}^{z} (z - \zeta)^{r-1} \left[f^{(r)}(\zeta) - P_n^{(r)}(\zeta) \right] d\zeta \right| \\ &\leqslant \frac{c}{n^r} \varphi^{r - \beta}(z) + c \left| \int_{z_j}^{z} (z - \zeta)^{r-1} P_n^{(r)}(\zeta)\, d\zeta \right| \\ &\leqslant \frac{c}{n^r} \varphi^{r - \beta}(z). \end{aligned}$$

So we have not yet proved (5.10) for the case $1 \leqslant \beta < 2$, $\alpha_j > 1$ and $\left|z - z_j\right| \leqslant n^{-\alpha_j}$. In this case we may apply the same arguments as for the case $\beta \geqslant 2$, but then we will obtain a constant $\frac{c}{2-\beta}$, which depends on β. Till this moment in the proof of Theorem 7 we had constants independent of β for free. Therefore there is a reason to give arguments that eliminate the dependence of a constant on β in this case as well. To this end, we take a point $z_0 \in G$, satisfying $\left|z_0 - z_j\right| = n^{-\alpha_j}$. Since $1 \leqslant \beta < 2$, then $T^{r-[\beta]} \equiv f^{(r-1)}(z_0)$. We represent (5.12) in the form

$$\begin{aligned} &((r-2)!)\,(f(z) - P_n(z)) \\ &\quad = \int_{z_j}^{z} (z-\zeta)^{r-2} \left(f^{(r-1)}(\zeta) - f^{(r-1)}(z_0) \right) d\zeta \\ &\qquad - \int_{z_j}^{z} (z-\zeta)^{r-2} \left(R_n^{(r-1)}(\zeta) - T^{(r-1)}(z_0,\zeta) + \frac{1}{n^\beta}\, Q_n^{(r-1)}(\zeta) \right) d\zeta \\ &\quad =: \mu_1 - \mu_2. \end{aligned}$$

For μ_2 estimates (5.6) and (4.6) yield $\left|\mu_2\right| \leqslant \frac{c}{n^r} \varphi^{r-\beta}(z)$. So we have to estimate μ_1. Denote by γ a proper curve, guaranteed by (b) of Lemma 3 for the points $z^0 := z$ and z_0. Let $\zeta(s)$ be its natural parametrization, $z_j = \zeta(0)$, $z = \zeta(s^0)$ and $z_0 = \zeta(s_0)$. Let γ_s be the arc of γ with the endpoints $\zeta(s)$ and z_0. To avoid too much writing we put $\sigma := \frac{\beta}{\alpha_j} - \beta$ and note that $0 > \sigma \geqslant -\frac{\beta}{2} > -1$, and for all $\zeta \in \gamma \setminus \{z_j\}$ we have $\left|\zeta - z_j\right|^\sigma \leqslant c\varphi^{-\beta}(\zeta) \leqslant c\left|\zeta - z_j\right|^\sigma$, whence $\left|f^{(r)}(\zeta)\right| \leqslant c\left|\zeta - z_j\right|^\sigma \leqslant cs^\sigma$. Now, if $s^0 < s_0$, then

$$\left|f^{(r-1)}(\zeta) - f^{(r-1)}(z_0)\right| = \left|\int_\zeta^{z_0} f^{(r)}(\zeta) d\zeta\right| \leqslant c \int_s^{s_0} u^\sigma\, du.$$

Since

$$\int_0^{s^0} \left(\int_s^{s_0} u^\sigma\, du \right) ds \leqslant s_0 (s^0)^{1+\sigma},$$

then

$$\begin{aligned} \left|\mu_1\right| &\leqslant \int_\gamma |z-\zeta|^{r-2} \left| \int_\zeta^{z_0} f^{(r)}(\xi)\, d\xi \right| |d\zeta| \\ &\leqslant c \int_0^{s^0} (s^0 - s)^{r-2} \left(\int_s^{s_0} u^\sigma\, du \right) ds \\ &\leqslant c (s^0)^{r-2} \int_0^{s^0} \left(\int_s^{s_0} u^\sigma\, du \right) ds \\ &\leqslant c s_0 (s^0)^{r-1+\sigma} \leqslant c \left|z_0 - z_j\right| \left|z - z_j\right|^{r-1+\sigma} \leqslant \frac{c}{n^r}\, \varphi^{r-\beta}(z). \end{aligned}$$

Finally, if $s^0 > s_0$, then $\left|\zeta - z_j\right| \geqslant c\left|z_0 - z_j\right|$ for all $\zeta \in \gamma_{s^0}$, and the required estimate is evident. Theorem 7 is proved. □

Proof of Theorem 2. If $n \geqslant rl/\alpha$, then Theorem 2 follows from Theorem 7. On the other hand in the proof of Theorem 2 we do not have to pay attention to z_j's with $\alpha_j < 1$.

Therefore in all our arguments above one can take $r^* = r$ and thus obtain Theorem 2 for all $n \geqslant lr$. $\square$

Note that, the same arguments provide the validity of Theorem, formulated in the Abstract to the paper.

Proof of Theorem 4. If $n \geqslant rl/\alpha$, then Theorem 4 is a particular case of Theorem 7. If $n < rl/\alpha$, then $c \leqslant \rho_n(z) \leqslant c$, $z \in G$. Now we take $P_n(z) := T(\hat{z}, z)$ ($\equiv 0$ by our assumption (4.1)). Then (1.3) follows from Lemma 4, and (1.4) is trivial. $\square$

6. Inverse theorems

We begin with

Proof of Theorem 8. If the right-hand side of (1.8) is equal to $+\infty$, then there is nothing to prove. So we assume that there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of numbers $n_k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \left(a_{n_k} + b_{n_k}\right) = 1,$$

where

$$a_n := r!n^r \left\| \frac{(f - P_n)\varphi^\beta}{\varphi^r} \right\|_G \quad \text{and} \quad b_n := \left\| P_n^{(r)} \varphi^\beta \right\|_G .$$

We fix $z \in G$, put $U_n := \left\{\zeta \in \mathbb{C}\colon\ |\zeta - z| \leqslant \frac{1}{n}\varphi(z)\right\}$ and note that if $U_n \subset G$, then

$$\max_{\zeta\in U_n} |f(\zeta) - P_n(\zeta)| \leqslant \frac{a_n}{r!n^r} \max_{\zeta\in U_n} \varphi^{r-\beta}(\zeta).$$

Therefore the integral Cauchy formula implies

$$\left| f^{(r)}(z) - P_n^{(r)}(z) \right| = \frac{r!}{2\pi} \left| \int_{\partial U_n} \frac{f(\zeta) - P_n(\zeta)}{(\zeta - z)^{r+1}}\, d\zeta \right| \leqslant \frac{a_n}{\varphi^r(z)} \max_{\zeta\in U_n} \varphi^{r-\beta}(\zeta).$$

Now we take K so large that $U_{n_k} \subset G$ for all $k > K$. Then, for all $k > K$, we obtain

$$\begin{aligned} \left| f^{(r)}(z) \right| \varphi^\beta(z) &\leqslant \left| f^{(r)}(z) - P_{n_k}^{(r)}(z) \right| \varphi^\beta(z) + \left| P_{n_k}^{(r)}(z) \right| \varphi^\beta(z) \\ &\leqslant \frac{a_{n_k}}{\varphi^{r-\beta}(z)} \max_{\zeta\in U_{n_k}} \varphi^{r-\beta}(\zeta) + b_{n_k} \to 1, \quad n \to \infty. \end{aligned}$$

The theorem is proved. $\square$

Note that, the above proof shows that Theorem 8 is valid for any open set $\widetilde{G}$ and each continuous and positive on $\widetilde{G}$ function $\widetilde{\varphi}$, instead of G and φ, respectively.

To prove Theorems 3 and 6 we need

Lemma 9. *Let $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$. If a function f is continuous on $\overline{G}$ and analytic in G, then*

$$\left\| f\rho_n^{\gamma} \right\|_G \leqslant C(G,\gamma) \left\| f\rho_n^{\gamma} \right\|_{\partial G}.$$

Proof. For each $j = 1, \dots, l$ we denote by $\tilde{z}_j$ a point, satisfying $\tilde{z}_j \notin \overline{G}$, $\left|\tilde{z}_j - z_j\right| = \rho_n(z_j)$, and $\left|z - \tilde{z}_j\right| \geqslant c \left|z - z_j\right|$ for all $z \in \overline{G}$. Since $\alpha \neq 0$, then such a point exists. Then, for all $z \in \overline{G}$ we have

$$\rho_n(z) \leqslant \frac{c}{n} \prod_{j=1}^{l} \left|z - \tilde{z}_j\right|^{1-\frac{1}{\alpha_j}} \leqslant c\rho_n(z).$$

Therefore it is sufficient to prove the inequality

$$\|f\Pi\|_G \leqslant \|f\Pi\|_{\partial G}, \tag{6.1}$$

where

$$\Pi(z) = \prod_{j=1}^{l} \left|z - \tilde{z}_j\right|^{\beta_j} \text{ and } \beta_j = \gamma\left(1 - \frac{1}{\alpha_j}\right).$$

Now, if all β_j's are rational numbers, say $\beta_j = p_j/q_j$, then (6.1) is equivalent to

$$\left\| f^q \pi^q \right\|_G \leqslant \left\| f^q \pi^q \right\|_{\partial G},$$

where $q = q_1 \cdot \ldots \cdot q_l$, which is evident, since $|f^q \pi^q|$ is a modulus of the analytic in G function $f^q(z) \prod_{j=1}^{l} \left(z - \tilde{z}_j\right)^{p_j}$. If not all β_j's are rational numbers, then we take a sequence $\{\beta_1^{(n)}, \dots, \beta_l^{(n)}\}_{n=1}^{\infty}$ of vectors with rational coordinates $\beta_j^{(n)}$, such that $(\beta_1^{(n)}, \dots, \beta_l^{(n)}) \to (\beta_1, \dots, \beta_l)$, $n \to \infty$, and obtain (6.1) by a passage to the limit. The lemma is proved. □

Having Lemma 9, one may rewrite Dzjadyk inequality (2.9) in the form

$$\left\| P_n' \rho^{\gamma+1} \right\|_G \leqslant C(\gamma, G) \left\| P_n \rho^{\gamma} \right\|_G, \tag{6.2}$$

for each $P_n \in \mathbb{P}_n$.

Now Theorem 3 readily follows from (6.2) and the expansion of f in Bernstein telescope series

$$f = P_{2^{k_0}} + \sum_{k=k_0}^{\infty} \left(P_{2^{k+1}} - P_{2^k}\right). \tag{6.3}$$

The same concerns Theorem 6, if one takes into account Lemma 10 below. Anyway, for completeness we prove both Theorems.

Proof of Theorem 3. To avoid too much writings we will write $\|\cdot\|$ instead of $\|\cdot\|_G$. If the right-hand side of (1.2) is infinity, then there is nothing to prove. So we assume that

$\|f - P_n\| \leqslant n^{-r-\varepsilon}$ for all $n \geqslant r$. Since by the assumption of Theorem 3, $\alpha_j \geqslant 1$ for all $j = 1, \dots l$, then $\rho_n(z) \geqslant \frac{c}{n}\varphi(z)$, $z \in G$. Therefore (6.2) yields, for all $n \geqslant r$,

$$\left\|(P_{2n}^{(r)} - P_n^{(r)})\varphi^r\right\| \leqslant cn^r \left\|(P_{2n}^{(r)} - P_n^{(r)})\rho_n^r\right\| \leqslant cn^r \|P_{2n} - P_n\| \\ \leqslant cn^r \|P_{2n} - f\| + cn^r \|P_n - f\| \leqslant cn^{-\varepsilon}.$$

Let $2^{k_0-1} < r \leqslant 2^{k_0}$. Then (6.3) implies

$$\left\|f^{(r)}\varphi^r\right\| \leqslant \left\|P_{2^{k_0}}^{(r)}\varphi^r\right\| + c\sum_{k=k_0}^{\infty}\frac{1}{2^{\varepsilon k}} \leqslant \left\|P_{2^{k_0}}^{(r)}\varphi^r\right\| + \frac{c}{\varepsilon},$$

which simultaneously guarantees the convergence in G to f of the Bernstein series and its derivatives. Since $P_r^{(r)} \equiv 0$ then

$$\left\|P_{2^{k_0}}^{(r)}\varphi^r\right\| = \left\|(P_{2^{k_0}}^{(r)} - P_r^{(r)})\varphi^r\right\| \leqslant c.$$

Thus, $\left\|f^{(r)}\varphi^r\right\| \leqslant c/\varepsilon$. The theorem is proved. □

Lemma 10. *For each polynomial $P_n \in \mathbb{P}_n$ we have*

$$\left\|\frac{P_n}{\rho_n^r}\right\|_G \leqslant cn^r \left\|\frac{P_n}{\varphi^r}\right\|_G. \tag{6.4}$$

Proof. Assume that $a := n^r \left\|P_n\varphi^{-r}\right\|_G$ is a bounded number. Let $c_1 \geqslant 1$ be a constant, defined by Lemma 2, $c_2 \geqslant 1$ be a constant, defined by the inequality (6.2) for $\gamma = r$, and $c_3 := (4c_1c_2)^{-1}$. Denote $D_j := \left\{\zeta \in \overline{G} : \left|\zeta - z_j\right| \leqslant c_3 n^{-\alpha_j}\right\}$, and $D := \bigcup_{j=1}^{l} D_j$. By Definition 2 of $\rho_n(z)$, $\varphi(z) \leqslant c\rho_n(z) \leqslant c\varphi(z)$, $\quad z \in \overline{G} \setminus D$, whence

$$\left\|P_n\rho_n^{-r}\right\|_{\overline{G}\setminus D} \leqslant c_4 a. \tag{6.5}$$

Put

$$A := \frac{1}{a}\left\|P_n\rho_n^{-r}\right\|_{\overline{G}}.$$

Assume that $A > c_4$. Then there is j_* and a point $z \in D_{j_*}$, such that

$$|P_n(z)| = Aa\rho_n^r(z) = Aa(n^{-\alpha_{j*}})^r =: Aa\rho_*^r.$$

Denote by $z_0 \in \overline{G}$ a point, such that $\left|z_0 - z_{j_*}\right| = c_3\rho_*$, and let $\gamma = \gamma(z, z_0)$ be a proper curve, guaranteed by Lemma 2. Since $|z - z_0| \leqslant 2c_3\rho_*$, one has *diam* $\gamma \leqslant c_1(2c_3)\rho_*$, and, therefore, for $\zeta \in \gamma$, whence

$$\left|\zeta - z_{j_*}\right| \leqslant |\zeta - z_0| + \left|z_0 - z_{j_*}\right| \leqslant (1 + 2c_1)c_3\rho_* \leqslant \rho_*,$$

hence by Definition 2 of ρ_n,

$$\rho_n(\zeta) = \rho_*, \quad \zeta \in \gamma.$$

Note that (6.5) implies

$$|P_n(z_0)| \leqslant c_4 a \rho_*^r.$$

Therefore, applying (6.2), we obtain

$$\begin{aligned} Aa\rho_*^r = |P_n(z)| &\leqslant |P_n(z_0)| + \left| \int_{z_0}^{z} P_n'(\zeta)\, d\zeta \right| \\ &\leqslant c_4 a \rho_*^r + c_1 |z - z_0| \max_{\zeta \in \gamma} \left| P_n'(\zeta) \right| \leqslant c_4 a \rho_*^r + 2c_1 c_3 \rho_* c_2 \rho_*^{r-1} Aa \\ &\leqslant c_4 a \rho_*^r + \frac{1}{2} Aa \rho_*^r, \end{aligned}$$

whence $A \leqslant 2c_4$. This implies (6.4) with $c = 2c_4$. □

Now we are ready to prove Theorem 6.

Proof of Theorem 6. To avoid too much writings we will write $\|\cdot\|$ instead of $\|\cdot\|_G$. If the right-hand side of (1.5) is infinity, then there is nothing to prove. So we assume that $\left\|(f - P_n)\varphi^{-r}\right\| \leqslant n^{-r-\varepsilon}$ for all $n \geqslant lr$. Then (6.2) yields, for all $n \geqslant lr$,

$$\left\| P_{2n}^{(r)} - P_n^{(r)} \right\| \leqslant c \left\| (P_{2n} - P_n)\rho_n^{-r} \right\| \leqslant cn^r \left\| (P_{2n} - P_n)\varphi^{-r} \right\|,$$

where we used Lemma 10 in the last inequality. Therefore

$$\left\| P_{2n}^{(r)} - P_n^{(r)} \right\| \leqslant cn^r \left\| (P_{2n} - f)\varphi^{-r} \right\| + cn^r \left\| (f - P_n)\varphi^{-r} \right\| \leqslant cn^{-\varepsilon}.$$

Let $2^{k_0 - 1} < lr \leqslant 2^{k_0}$. Then (6.3) implies

$$\left\| f^{(r)} \right\| \leqslant \left\| P_{2^{k_0}}^{(r)} \right\| + c \sum_{k=k_0}^{\infty} \frac{1}{2^{\varepsilon k}} \leqslant \left\| P_{2^{k_0}}^{(r)} \right\| + \frac{c}{\varepsilon},$$

which simultaneously guarantees the convergence in G to f of the Bernstein series and its derivatives. Moreover, this yields that f is continuous on $\overline{G}$. Therefore $E_r := E_r(f, G) < \infty$, and there exists the polynomial $P_r \in \mathbb{P}_r$ of the best approximation of f on $\overline{G}$, and hence on G. Since $P_r^{(r)} \equiv 0$, then

$$\begin{aligned} \left\| P_{2^{k_0}}^{(r)} \right\| = \left\| P_{2^{k_0}}^{(r)} - P_r^{(r)} \right\| &\leqslant c \left\| (P_{2^{k_0}} - P_r)\rho_{2^{k_0}}^{-r} \right\| \\ &\leqslant c \left\| (P_{2^{k_0}} - f)\rho_{2^k}^{-r} \right\| + c \left\| P_r - f \right\| \\ &\leqslant c \left\| (P_{2^{k_0}} - f)\varphi^{-r} \right\| + cE_r \leqslant c + cE_r, \end{aligned}$$

where we again applied inequality (6.2) and Lemma 10. Thus, $\left\| f^{(r)} \right\| \leqslant cE_r + c/\varepsilon$. The theorem is proved. □

References

[1] F.G. Abdullayev, Uniform convergence of the Generalized Bieberbach Polynomials in regions with non-zero angles, Acta Math. Hung. 77 (3) (1997) 223–246.

[2] G.A. Alibekov, On the properties of conformal mapping of the regions with corners, in: Voprosy Teorii Priblizhenij Funcij i jeje Prilozhemij, Inst. of Math. AN UkSSR, Kyiv, 1976, pp. 4–18, (in Russian).

[3] V.V. Andrievskii, V.I. Belyi, V.K. Dzjadyk, Conformal Invariants in Constructive Theory of Functions of Complex Variable, World Federation Publisher, Georgia, 1995.

[4] K.I. Babenko, On some problems of the approximation theory and numerical analysis, Uspekhi Math. Nauk 40 (1) (1985) 3–28 (in Russian).

[5] R.A. DeVore, Degree of approximation, in: G.G. Lorentz, C.K. Chui, L.L. Schumaker (Eds.), Approximation Theory II, Proceedings of the International Conference on Approximation Theory, Austin, Academic Press, New York, 1976, pp. 117–161.

[6] R.A. DeVore, Pointwise approximation by polynomials and splines, in: S.B. Stechkin, S.A. Teljakovskiĭ (Eds.), The Theory of Approximation of Functions, Proceedings of the International Conference, Kaluga, Nauka, Moskow, 1977, pp. 132–144.

[7] Z. Ditzian, D. Jiang, Approximation of functions by polynomials in $C[-1, 1]$, Canad J. Math. 4 (1992) 924 –940.

[8] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer Series in Computational Mathematics, Springer, New York, 1987.

[9] V.K. Dzjadyk, Introduction in Theory of Uniform Approximation of Function by Polynomials, Nauka, Moskow, 1977 (in Russian).

[10] H.H. Gonska, E. Hinnemann, Punktweise Absch atzungen zur Approximation durch alge braische Polynome, Acta Math. Hung. 16 (1985) 243–254.

[11] H.H. Gonska, D. Leviatan, I.A. Shevchuk, H.J. Wenz, Interpolatory pointwise estimates for polynomial approximation, Constr. Approx. 16 (4) (2000) 603–629.

[12] I.E. Gopengauz, A theorem of A.F.Timan on the approximation of functions by polynomials on a finite segment, Mat. Zametki 1 (1967) 163–172 (in Russian).

[13] E. Hinnemann, H.H. Gonska, Generalization of the theorem of DeVore, in: C.K. Chui et al. (Eds.), Approximation Theory IV, Proceedings of the International Symposium on Approximate Theory, College Station 1983, Academic Press, New York, NY, 1983, pp. 527–532.

[14] Ch. Pommerenke, Boundary Behavior of Conformal Maps, Springer, Berlin, 1992.

[15] I.A. Shevchuk, Approximation by polynomials and traces of functions continuous on a segment, Naukova Dumka, Kyiv, 1992 (in Russian).

[16] P.M. Tamrazov, Smoothnesses and Polynomial Approximations, Naukova Dumka, Kyiv, 1975 (in Russian).

[17] S.A. Teljakovskiĭ, Two theorems on the approximation of functions by algebraic polynomials, Matem. Sbornik, N.S. 70 (1966) 252–265 (in Russian); English transl. in Amer. Math. Soc. Transl. Ser. 2 (77) (1968) 163–178.

[18] R.M. Trigub, Constructive characteristic of certain classes of functions, Izv. AS SSSR, Ser. Math. 29 (3) (1965) 615–630 (in Russian).

[19] M. Zamanski, Classes de saturation des procédés de sommation des séries de Fourier et applications des series trigonométriques, Ann. Sci. Ecole Norm. Sup. (3) 67 (1950) 161–198.